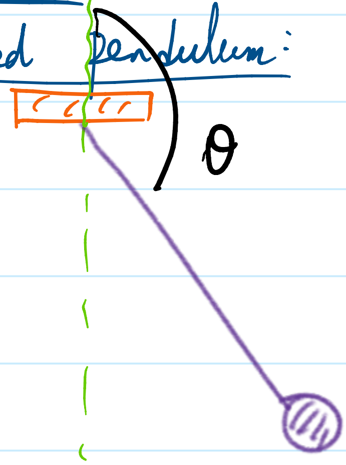
Lecture 21:

Damped pendulum:

22-11-18



We obtain the second order ODE on θ:

$$\theta'' + \underbrace{\gamma}_{\text{fraction term}} \theta' + \omega^2 \sin\theta = 0.$$

We turn it into a equivalent system:

by $\quad y_1 = \theta, \quad y_2 = \theta'$

$$\vec{y}\,'(t) = \begin{pmatrix} y_2 \\ -\omega^2 \sin y_1 - \gamma y_2 \end{pmatrix} := f(\vec{y}).$$

Step 1: Find all critical points

i.e. $\quad f(\vec{y}) = 0 \iff y_1 = k\pi, \; y_2 = 0.$

write $\vec{y}_k := \begin{pmatrix} k\pi \\ 0 \end{pmatrix}$

Step 2: Compute the associated linear system for each critical pt:

$$\left(\frac{\partial f}{\partial y}\right)_{ij} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos y_1 & -\gamma \end{pmatrix}$$

at $y_1 = k\pi, \; y_2 = 0 \Rightarrow \left(\frac{\partial f}{\partial y}\right) = \begin{pmatrix} 0 & 1 \\ \pm\omega^2 & -\gamma \end{pmatrix}$

$-$ if k even

$+$ if k odd

Step 3: For each critical pt $\vec{y}_*$, compute eigenvalues of $Df(\vec{y}_*)$:

Let $k = 2m\pi$:

$$\det(Df(\vec{y}_{2m}) - xI)$$

$$= \det\begin{pmatrix} -x & 1 \\ -\omega^2 & -\gamma - x \end{pmatrix} = x(x+\gamma) + \omega^2$$

$$= x^2 + \gamma x + \omega^2$$

$$r_1 = \frac{-\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - 4\omega^2}, \quad r_2 = \frac{-\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - 4\omega^2}$$

Cases:

a) $\gamma^2 > 4\omega^2$, $\quad r_1, r_2$ are distinct real negative eigenvalues

$\Longrightarrow$ A. stable node

b) $\gamma^2 = 4\omega^2$, $\quad$ repeated negative eigenvalue

$\longrightarrow$ A. stable (type undetermined)

c) $\gamma^2 < 4\omega^2$, $\quad$ complex pair of eigenvalues with $Re(r_1) = Re(r_2) < 0$

$\longrightarrow$ A. stable spiral

For $k=(2m+1)\pi$: $\quad Df(\vec{y}_k) = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}$

$$\det\left(Df(\vec{y}_k) - x I\right) = \det\begin{pmatrix} -x & 1 \\ \omega^2 & -\gamma - x \end{pmatrix}$$
$$= x^2 + \gamma x - \omega^2 .$$

$$r_1 = \frac{-\gamma}{2} + \frac{\sqrt{\gamma^2+4\omega^2}}{2}, \quad r_2 = \frac{-\gamma}{2} - \frac{\sqrt{\gamma^2+4\omega^2}}{2}.$$

$$r_2 < 0 < r_1$$

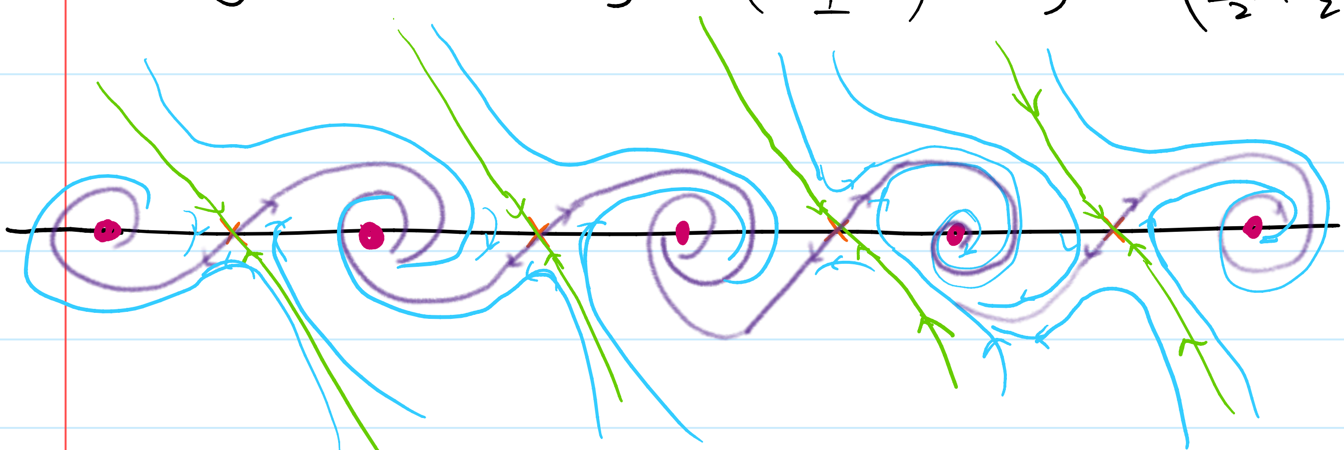
$\Longrightarrow$ we have an unstable saddle.

Step 4 : We will use this to draw the phase portrait.

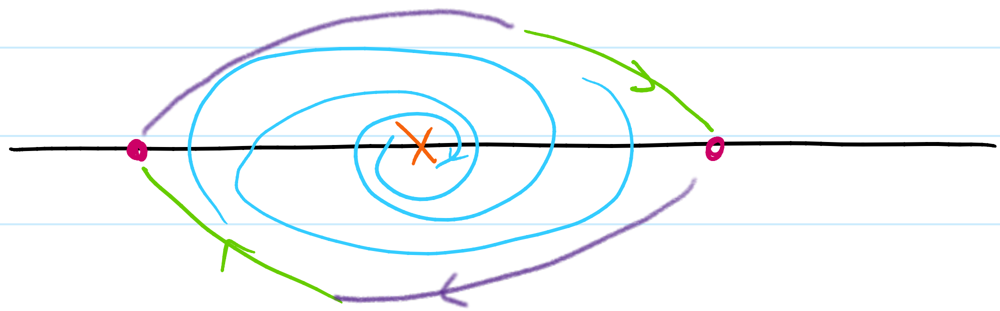
(Assume $\gamma = \omega = 1$)

For y_k s.t k odd : $\quad r_1 = \frac{-1}{2} + \frac{\sqrt{5}}{2}, \quad r_2 = \frac{-1}{2} - \frac{\sqrt{5}}{2}.$

eigenvectors: $\quad \xi_1 = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -1 \\ \frac{\sqrt{5}}{2} + \frac{1}{2} \end{pmatrix}$



Question: Why it cannot be



?

Method 1: numerically, we can plot enough vector field to determine which is the case

Method 2: consider $V(y_1, y_2) = 1 - \cos y_1 + \frac{1}{2} y_2^2$

total kinetic + potential energy for $\omega = 1$.

and show that $\frac{d}{dt}(V(\vec{y}(t))) < 0$ if $\gamma > 0$.